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Smoothed Particle Hydrodynamics (SPH) – Basics

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Smoothed Particle Hydrodynamics

Natural phenomena

PHYSICAL GOVERNING EQUATIONS

EULERIAN DESCRIPTION

$$\text{Eulerian derivative} = \frac{\partial}{\partial t}$$

MESHBASED METHODS

Finite-difference
Finite-element
Finite-volume

COMPUTATIONAL METHODS

LAGRANGIAN DESCRIPTION

$$\text{Lagrangian derivative} = \frac{\mathrm{d}}{\mathrm{d}t}$$

MESHFREE METHODS

SMOOTHED PARTICLE HYDRODYNAMICS

Meshless methods

Eulerian description of motion: describes changes as they occur at a fixed point in the fluid

Lagrangian description of motion: describes changes which occur as you follow a fluid particle along its trajectory

The Eulerian derivative is the rate of change at a fixed position

e.g. measuring the flow in a river at a fixed location

$$\text{Eulerian derivative} = \frac{\partial}{\partial t} \longrightarrow \boxed{\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}}$$

The Lagrangian derivative is the '**total** rate of change' and is the derivative along a fluid trajectory

$$\text{Lagrangian derivative} = \frac{\mathrm{d}}{\mathrm{d}t}$$

Mesh-based methods

In general:

Mesh-based methods are **good** for

(i) confined computational domains

(ii) computations where the boundaries are not moving

Mesh-based methods are “**bad”** for:

(i) mesh generation – can be very expensive

(ii) highly nonlinear deformation of the fluid body

➔ **So, we need numerical methods where we are not constrained by the restrictions of the numerical mesh**…. Hence, **Meshless methods**

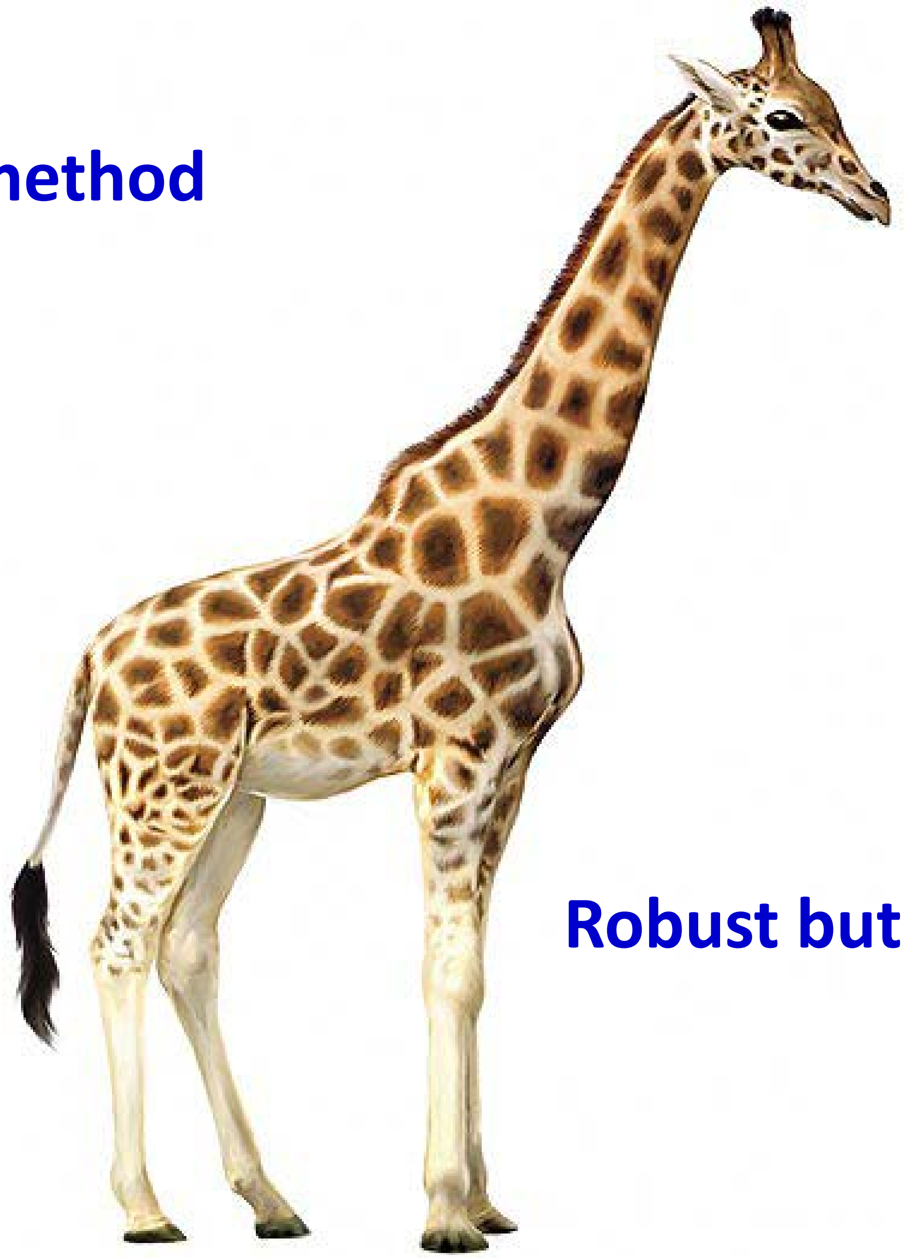
Meshless methods

Quick comparison of Lagrangian & Eulerian methods:

	Lagrangian Methods	**Eulerian Methods**
Grid	Attached to moving particles	Fixed in space
Track	Movement of any point on materials	Mass, momentum & energy flux across grid nodes & mesh cell boundary
Time history	Easy to obtain time-history on point attached to materials	Difficult to obtain time-history on point attached to materials
Moving boundaries & interfaces	Easy to track	Difficult to track
Irregular geometry	Easy to model	Difficult to model with accuracy

From Liu & Liu (2003)

Mesh-based method



Robust but rigid

Following L.D. Libertsky

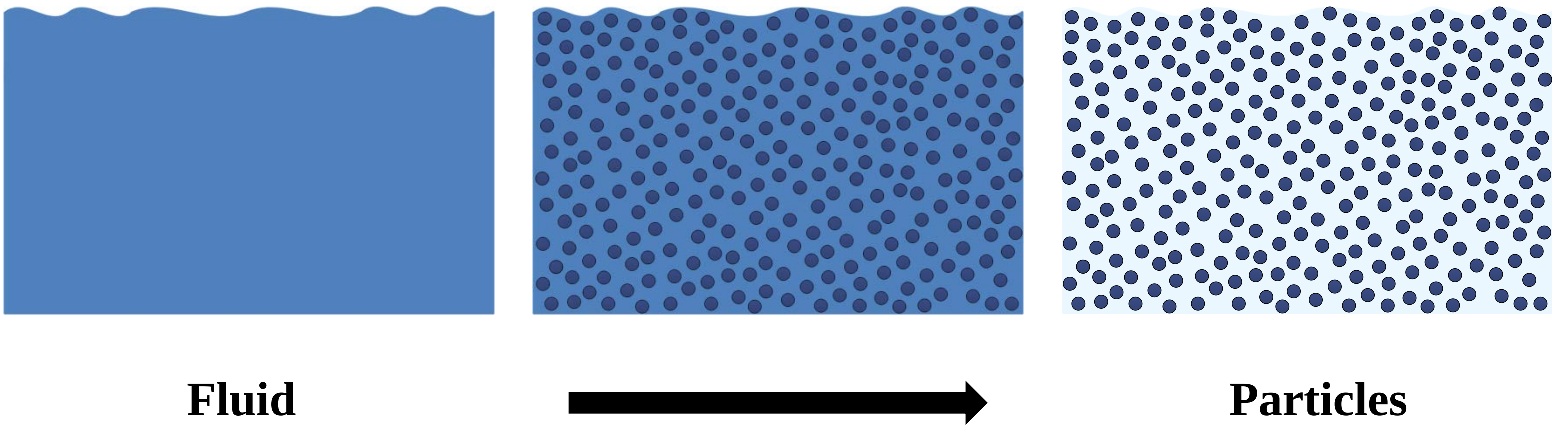
Mesh free

Flexible and adaptable

Meshless methods

- What is a **Meshless** method? **No computational grid or mesh**

The computational points now take the form of ‘particles’ or nodal interpolation ‘points’, similar (but different) to the nodal points in the Finite Element Method



Different meshless techniques are presented in **Lagrangian** form

Fundamentals of SPH

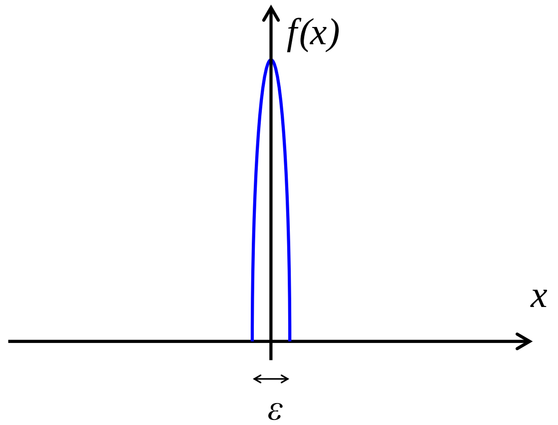
The Delta Function

The interpolation procedure within SPH is based on the (discrete) approximation that the value of a function $A(\mathbf{x})$ at a point $\mathbf{x}$ in space can be expressed as:

$$A(\mathbf{r}) = \int_{\Omega} \delta(\mathbf{r}-\mathbf{r}')\, A(\mathbf{r}')\, \mathrm{d}\,\Omega$$

where $\delta(\mathbf{x})$ is the Dirac delta function defined as

$$\delta_{\varepsilon}(x) = \lim_{\varepsilon\to 0} \begin{cases} 0 & x < -\varepsilon/2 \\ 1/\varepsilon & -\varepsilon/2 < x < \varepsilon/2 \\ 0 & x > \varepsilon/2 \end{cases}$$



Fundamentals of SPH

The SPH Integral Interpolation

In our computations, we cannot use a delta function since it is ***infinitesimally narrow*** which means that the interpolation region, Ω, would not overlap with other particles/nodal interpolation points.

Hence, the interpolation procedure within SPH approximates the delta function with its own weighting function called the SMOOTHING KERNEL, W

$$\langle A(\mathbf{r})\rangle = \int_{\Omega} W(\mathbf{r}-\mathbf{r}', h)\, A(\mathbf{r}')\,\mathrm{d}\Omega$$

where < · > is the integral SPH averaged quantity and

Fundamentals of SPH

The SPH Smoothing Kernel

As stated, the interpolation procedure within SPH approximates the delta function with its own weighting function called the SMOOTHING KERNEL

$$\langle A(\mathbf{r})\rangle = \int_{\Omega} W(\mathbf{r}-\mathbf{r}', h)\, A(\mathbf{r}')\,\mathrm{d}\,\Omega$$

The kernel depends on two quantities:

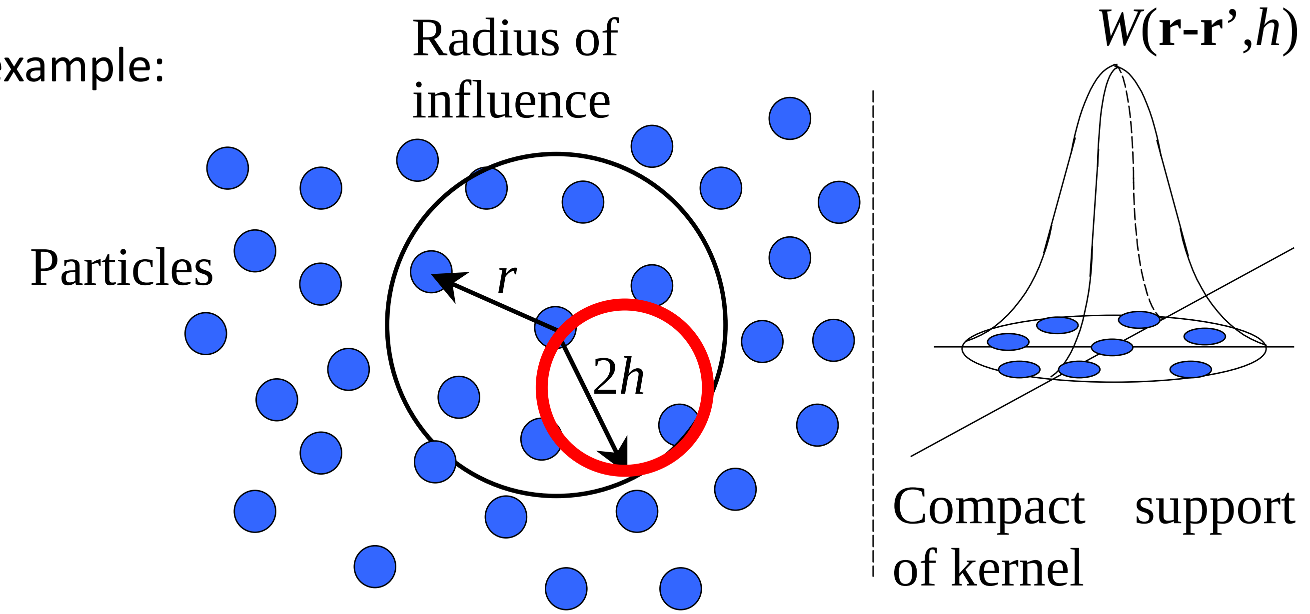
(i) The interpolation distance (distance between particles) = $\mathbf{r} - \mathbf{r}'$

(ii) The smoothing length, h (=characteristic length)

Fundamentals of SPH

The SPH Smoothing Kernel and smoothing length

Smoothing Kernel example:



The smoothing length *h* defines the extent of the kernel.

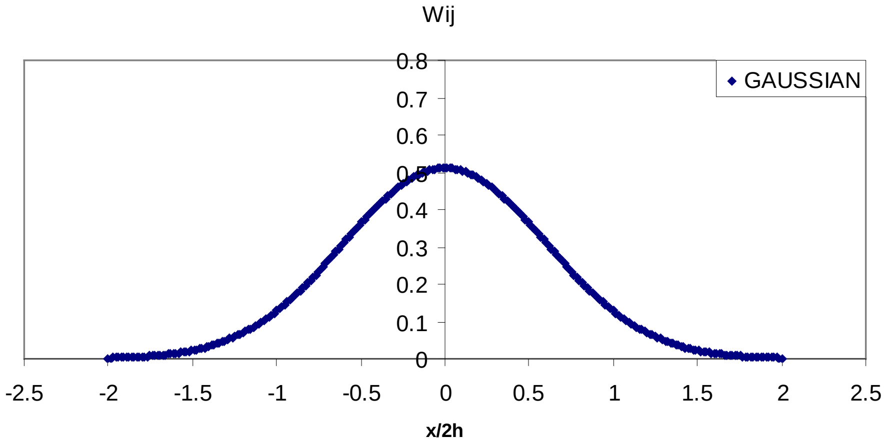
In SPH simulations, it is either:

(i) Kept constant at a present value

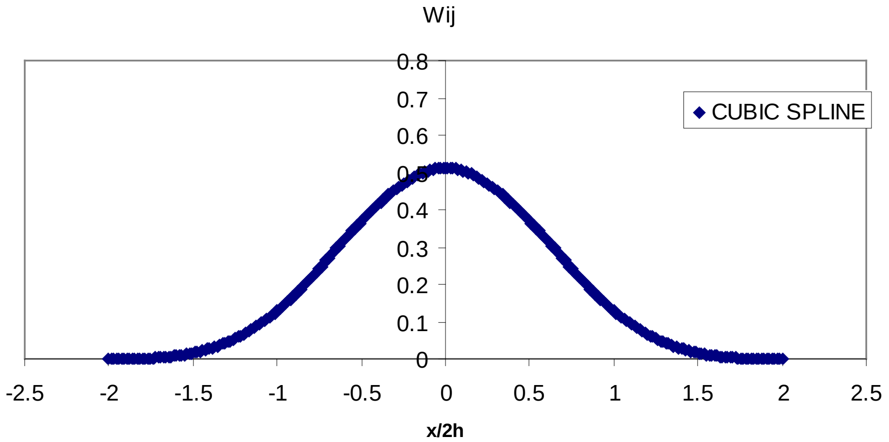
(ii) Adapted during the simulation according to some criterion (variable *h*)

Fundamentals of SPH

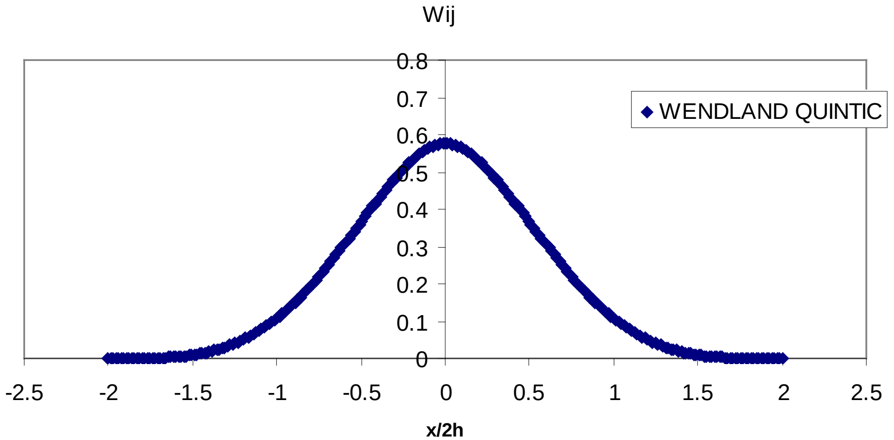
The SPH Smoothing Kernel



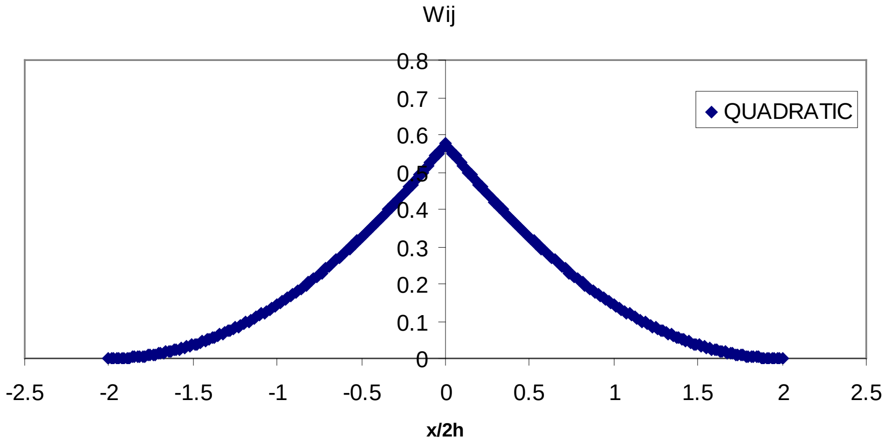
Gaussian



Cubic Spline (B-spline)



Higher-order kernels, e.g. 4th & 5th-order



Quadratic

Note symmetry of each kernel, and that the kernels =0 beyond $\pm 2h$

Fundamentals of SPH

The SPH Smoothing Kernel: Example 5th-order (Wendland kernel)

The Wendland smoothing kernel is defined as:

$$W(r,h) = \alpha_D \left(1 - \frac{q}{2}\right)^4 (2q+1) \qquad 0 \leq q \leq 2$$

α_D is normalisation factor to ensure integral of the kernel itself reproduces unity, and is defined as:

2-D: $7/(4\pi h^2)$

3-D: $7/(8\pi h^3)$

$$q = \frac{r - r'}{h}, \text{ or } q = \frac{r_i - r_j}{h}$$

Advantages: **(i) is high–order & therefore captures higher-order effects**

(ii) has improved accuracy

Disadvantage: **(i) is high–order & therefore computationally expensive**

(ii) has a point of maximum (extremum) in its gradient

Fundamentals of SPH

The DISCRETE SPH Interpolation procedure

In the numerical SPH method, we must approximate the integral interpolation procedure

$$A(\mathbf{r}) = \int_{\Omega} A(\mathbf{r}')W(\mathbf{r}-\mathbf{r}',h)\,\mathrm{d}\mathbf{r}'$$

$$\downarrow$$

$$A(\mathbf{r}) \approx \sum_{j=1}^{N} A(\mathbf{r}_j)W(\mathbf{r}-\mathbf{r}_j,h)\frac{m_j}{\rho_j}$$

where $\mathrm{d}\mathbf{r}'=\mathrm{d}\Omega$ becomes the volume of each particle

$$V_j = \frac{m_j}{\rho_j} = \frac{\text{mass of particle } j}{\text{density of particle } j}$$

Subscripts i or j denotes particles i or j

Fundamentals of SPH

Axioms of SPH Integral Interpolation

The interpolation procedure within SPH with the smoothing kernel depends on 3 axioms in order to give accurate results

Partition of unity:

$$(i) \qquad \int_{\Omega} W(\mathbf{r}-\mathbf{r}', h)\, \mathrm{d}\Omega = 1$$

Kernel tends to delta fn:

$$(ii) \qquad W(\mathbf{r}-\mathbf{r}', h) \to \delta(\mathbf{r}-\mathbf{r}'), \qquad h \to 0$$

Kernel is *k*-times differentiable

and its derivatives are continuous:

$$(iii) \qquad W(\mathbf{r}'-\mathbf{r}, h) \in C_0^k$$

SPH Equations

The governing equations we want to solve are the Navier-Stokes equations expressed in **Lagrangian form**:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho\nabla.\mathbf{v}$$

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\frac{1}{\rho}\nabla p + \upsilon_o\nabla^2\mathbf{u} + \mathbf{F}$$

We will try inviscid only

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho\nabla.\mathbf{v}$$

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\frac{1}{\rho}\nabla p + \mathbf{F}$$

ρ is density, $\mathbf{v}$ is velocity vector, t is time, p is pressure, v_0 is viscosity and $\mathbf{F}$ are body forces

SPH Continuity Equation

The continuity equation we wish to solve is:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho \cdot \nabla \mathbf{u} \quad \longrightarrow \quad \left\langle \frac{\mathrm{d}\rho}{\mathrm{d}t} \right\rangle = -\int_{-\infty}^{+\infty} \rho(x') \frac{\partial u(x')}{\partial x'} W(x - x') \,\mathrm{d}x'$$

After some algebra

$$\left\langle \frac{\mathrm{d}\rho}{\mathrm{d}t} \right\rangle = \sum_j m_j \left(\mathbf{u}_i - \mathbf{u}_j\right) \cdot \nabla_i W_{ij}$$

More generally for any function A, we can use the same procedure to write

$$\langle \nabla A \rangle = \sum_j \left(A_j - A_i\right) \nabla_i W_{ij} \frac{m_j}{\rho_j}$$

The Basic SPH Equations

$$\boxed{\begin{aligned} \frac{\mathrm{d}\rho}{\mathrm{d}t} &= -\rho\nabla.\mathbf{v} \\ \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} &= -\frac{1}{\rho}\nabla p + \mathbf{F} \end{aligned}}$$

$$\left\langle \frac{\mathrm{d}\rho}{\mathrm{d}t} \right\rangle = \sum_j m_j \left(\mathbf{v}_i - \mathbf{v}_j\right) \cdot \nabla_i W_{ij}$$

Conservation of Mass

$$\left\langle \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \right\rangle = \sum_j m_j \left(\frac{p_j}{\rho_j^2} + \frac{p_i}{\rho_i^2} \right) \nabla_i W_{ij}$$

Conservation of Momentum

$$\left\langle \frac{\mathrm{d}e}{\mathrm{d}t} \right\rangle = \frac{1}{2} \sum_j m_j \left(\frac{p_j}{\rho_j^2} + \frac{p_i}{\rho_i^2} \right) \mathbf{v}_{ij} \cdot \nabla_i W_{ij}$$

Conservation of Energy

These are the standard SPH equations used to solve many problems in the fields mentioned earlier (astrophysics, coastal hydrodynamics, gas dynamics, etc.)

We need one more equation: $$\frac{\mathrm{d}m_i}{\mathrm{d}t} = 0$$

So, the mass of each equation does not change (in the standard formulation)

Equation of state

We now examine how to close the equations for water using an equation of state.

We have a choice of 3 options: 2 different equations of state and 1 methodological.

(a) Tait's equation of state is

$$p = \frac{c_o^2 \rho_w}{\gamma}\left(\left(\frac{\rho}{\rho_w}\right)^{\gamma} - 1\right)$$

ρ_w = 1000 kg/m^3 density of water,

γ = 7 polytropic index,

c_o is the speed of sound for $\rho = \rho_w$

Note speed of sound for each particle $c_i = \sqrt{\frac{\partial P}{\partial \rho}}$

The water is **weakly compressible**.

(b) Morris's equation of state is (γ = 1)

$$p = c_o^2 \left(\rho - \rho_w\right)$$

ρ_w = = 1000 kg/m^3 density of water,

c_o is the speed of sound for $\rho = \rho_w$

The water is **more compressible**.

(c) Enforce incompressibility via the pressure Poisson equation

$$\nabla^2 p = ??$$

different forms of this Poisson equation

This is not very popular since it can be very **difficult** to maintain **strict incompressibility** and there are problems with boundary conditions.

Equation of state

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Fundamentals of Smoothed Particle Hydrodynamics (SPH)

Viscosity

SPH VISCOSITY

Viscous effects can be included in SPH in 3 ways:

(i) Artificial viscosity

(ii) Laminar viscosity

(iii) Turbulence models

(i) Artificial viscosity

As the name suggests, this is artificial and uses empirical coefficients to model the energy dissipation.

In SPH notation, the momentum equation is written as

$$\boxed{\frac{d\mathbf{v}_a}{dt} = -\sum_b m_b \left(\frac{P_b}{\rho_b^2} + \frac{P_a}{\rho_a^2} + \Pi_{ab} \right) \nabla_a W_{ab} + \mathbf{g}}$$

SPH VISCOSITY

$$\frac{d\mathbf{v}_a}{dt} = -\sum_b m_b \left(\frac{P_b}{\rho_b^2} + \frac{P_a}{\rho_a^2} + \Pi_{ab} \right) \nabla_a W_{ab} + \mathbf{g}$$

The artificial viscosity proposed by Monaghan (1992) has been used very often due to its simplicity. In the limit Π_{ij} the viscosity term tends to the differential

$$\Pi_{ab} = \begin{cases} \dfrac{-\alpha \overline{c_{ab}} \mu_{ab}}{\overline{\rho_{ab}}} & \mathbf{v}_{ab} \cdot \mathbf{r}_{ab} < 0 \\ 0 & \mathbf{v}_{ab} \cdot \mathbf{r}_{ab} > 0 \end{cases}$$

$$\mu_{ab} = \frac{h \mathbf{v}_{ab} \cdot \mathbf{r}_{ab}}{\mathbf{r}_{ab}^2 + \eta^2}$$

$$\mathbf{r}_{ab} = \mathbf{r}_a - \mathbf{r}_b$$

$$\mathbf{v}_{ab} = \mathbf{v}_a - \mathbf{v}_b$$

$$\overline{\mathrm{c}_{\mathrm{a}}} =_{\mathrm{b}} \frac{\mathrm{c}_{\mathrm{a}} + \mathrm{c}_{\mathrm{b}}}{2}$$

$\eta^2 = 0.01h^2$

α is a free parameter that can be changed according to each problem.

Therefore it's empirical!! **BAD!**

SPH VISCOSITY

(ii) Laminar viscosity

The momentum conservation equation with laminar viscous stresses is given by

$$\frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho}\nabla P + \mathbf{g} + \upsilon_O \nabla^2 \mathbf{v}$$

where the laminar stress term simplifies (Morris et al., 1997) to

$$\left(\upsilon_0 \nabla^2 \mathbf{v}\right)_a = \sum_b m_b \left(\frac{4\upsilon_0 \mathbf{r}_{ab} \nabla_a W_{ab}}{(\rho_a + \rho_b)\left|\mathbf{r}_{ab}\right|^2} \right) \mathbf{v}_{ab}$$

where v_0 is the kinetic viscosity of laminar flow.

So, in SPH notation, the momentum equation becomes:

$$\boxed{\frac{\mathrm{d}\,\mathbf{v}_a}{\mathrm{d}\,t} = -\sum_b m_b \left(\frac{P_b}{\rho_b^2} + \frac{P_a}{\rho_a^2} \right) \nabla_a W_{ab} + \mathbf{g} + \sum_b m_b \left(\frac{4\upsilon_0 \mathbf{r}_{ab} \nabla_a W_{ab}}{(\rho_a + \rho_b)\left|\mathbf{r}_{ab}\right|^2} \right) \mathbf{v}_{ab}}$$

SPH VISCOSITY: Turbulence

(iii) Turbulence Modelling

This is **specialised** and really depends on what physics you are modelling and what level of sophistication you would want in your turbulence model.

Here we just give the governing equations we are solving and the corresponding SPH equations

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\frac{1}{\rho}\nabla P + \mathbf{g} + \upsilon_0 \nabla^2 \mathbf{v} + \frac{1}{\rho}\nabla \overset{=}{\tau}$$

τ represents the shear stresses due to turbulence.

The SPH form:

$$\boxed{\begin{aligned} \frac{\mathrm{d}\mathbf{v}_a}{\mathrm{d}t} &= -\sum_b m_b\left(\frac{P_b}{\rho_b^2} + \frac{P_a}{\rho_a^2}\right)\nabla_a W_{ab} + \mathbf{g} \\ &+ \underbrace{\sum_b m_b\left(\frac{4\upsilon_0 \mathbf{r}_{ab}\nabla_a W_{ab}}{(\rho_a+\rho_b)\left|r_{ab}\right|^2}\right)\mathbf{v}_{ab}}_{\text{Laminar}} + \underbrace{\sum_b m_b\left(\frac{\tau_b}{\rho_b^2} + \frac{\tau_a}{\rho_a^2}\right)\nabla_a W_{ab}}_{\text{Turbulent}} \end{aligned}}$$

Fundamentals of Smoothed Particle Hydrodynamics (SPH)

Corrections

Corrections

- **X SPH**
- **Tensile instability**

Corrections

- **X SPH**
- **Tensile instability.**

Increasing resolution

New kernels

Corrections

- **X SPH**
- **Tensile instability.**

Increasing resolution

New kernels

Kernel incomplete near boundaries or near the free surface

- **Kernel correction**
- **Kernel Gradient correction**

Corrections

- **X SPH** **Increasing resolution**
- **Tensile instability.** **New kernels**

Kernel incomplete near boundaries or near the free surface

- **Kernel correction** **Unnecessary**
- **Kernel Gradient correction** **Unstable**

Corrections

Density filters

- **Shepard filter (0 order)**
- **MLS (1st order)**

Others

- **Shifting**
- **Delta- SPH**

delta-SPH

new term in the **Continuity equation**

$$\frac{d\rho_a}{dt} = \sum_b m_b \mathbf{v}_{ab} \cdot \nabla_a W_{ab} + \boxed{2\delta_\Phi h c_0 \sum_b (\rho_b - \rho_a) \frac{\mathbf{r}_{ab} \cdot \nabla_a W_{ab}}{\mathbf{r}_{ab}^2} \frac{m_b}{\rho_b}}$$

The state equation describes a very stiff density field, and together with the natural disordering of the particles, high-frequency low amplitude oscillations are found to populate the density scalar field [Molteni and Colagrossi, 2009].

It’s a diffusive term to reduce density fluctuations

Fundamentals of Smoothed Particle Hydrodynamics (SPH)

Boundary conditions

Now, let's discuss boundaries in SPH.

When SPH was developed over 30 years ago, the technique was designed for **astrophysics simulating galaxy formation**, etc.

Up there (in space) **there are no boundaries**!! They didn't need to consider boundary conditions, but for engineering, all our simulations will have boundaries either **open or closed (solid wall).**

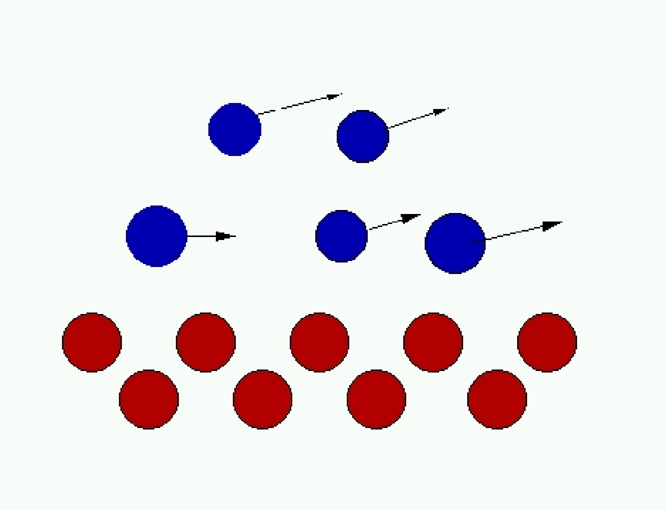
The immediate problem that presents itself is the old problem with the kernel- **there are no particles there**! A more philosophical Question is: If there are no fluid particles in the wall, **what role or function should any artificial particles take?**

Solid Wall Boundary Conditions

There are 3 basic choices:

(i) Fluid Particles do not move & remain still.

We calculate $\frac{\mathrm{d}\rho}{\mathrm{d}t}$, $\frac{\mathrm{d}e}{\mathrm{d}t}$ but $\mathbf{v}$ = 0 for all boundary particles



Advantages: **Simple**

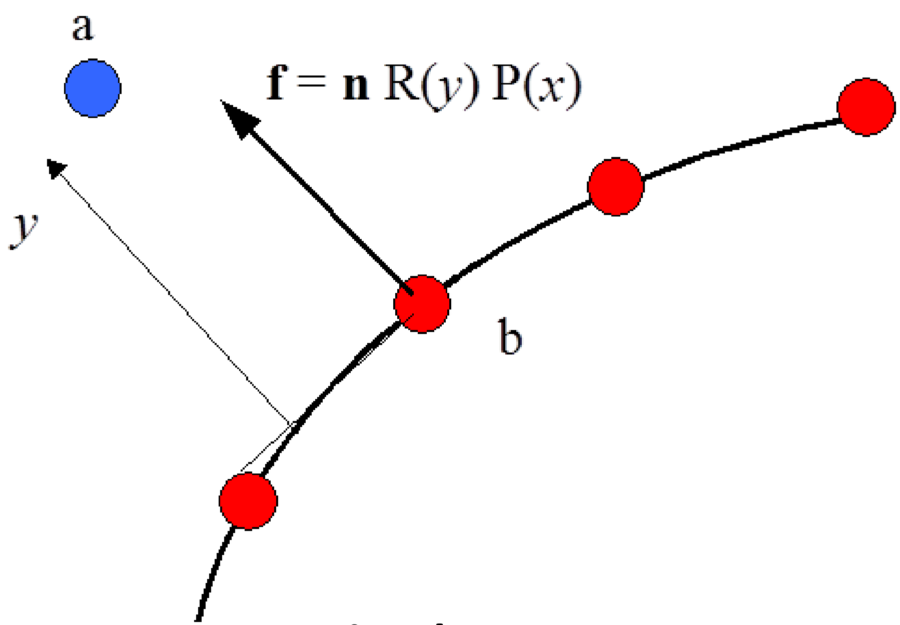
Complex geometries can be represented easily

Disadvantages: **Produces a very large Boundary Layer!**

Solid Wall Boundary Conditions

(ii) Repulsive Force

This can take various forms such as Lennard-Jones forces or an empirical function with a singularity so that the force increases as the particle nears the boundary



- Repulsive force calculation
- $R(y)$ is infinite as the normalized distance from the wall $y \to 0$

Advantages: **Simple**

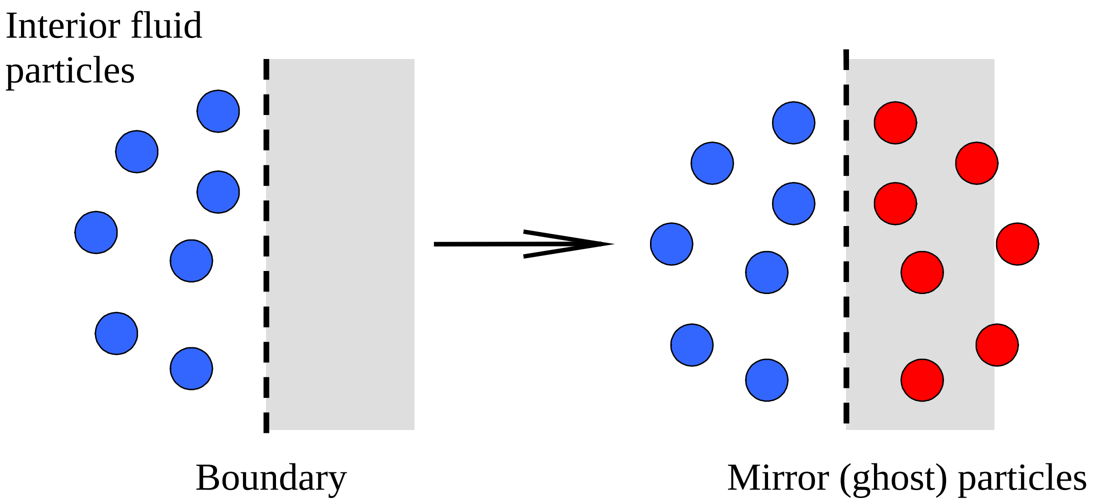
Virtually avoids wall penetration

Disadvantages: **Empirical!**

Solid Wall Boundary Conditions

(iii) Mirror Particles

When a real particle is close to a boundary then a virtual (ghost) particle is generated outside of the system, constituting the specular image of the incident one.



mirror particles have the same density and pressure, but opposite velocity

Advantages: **Most theoretically correct approach**

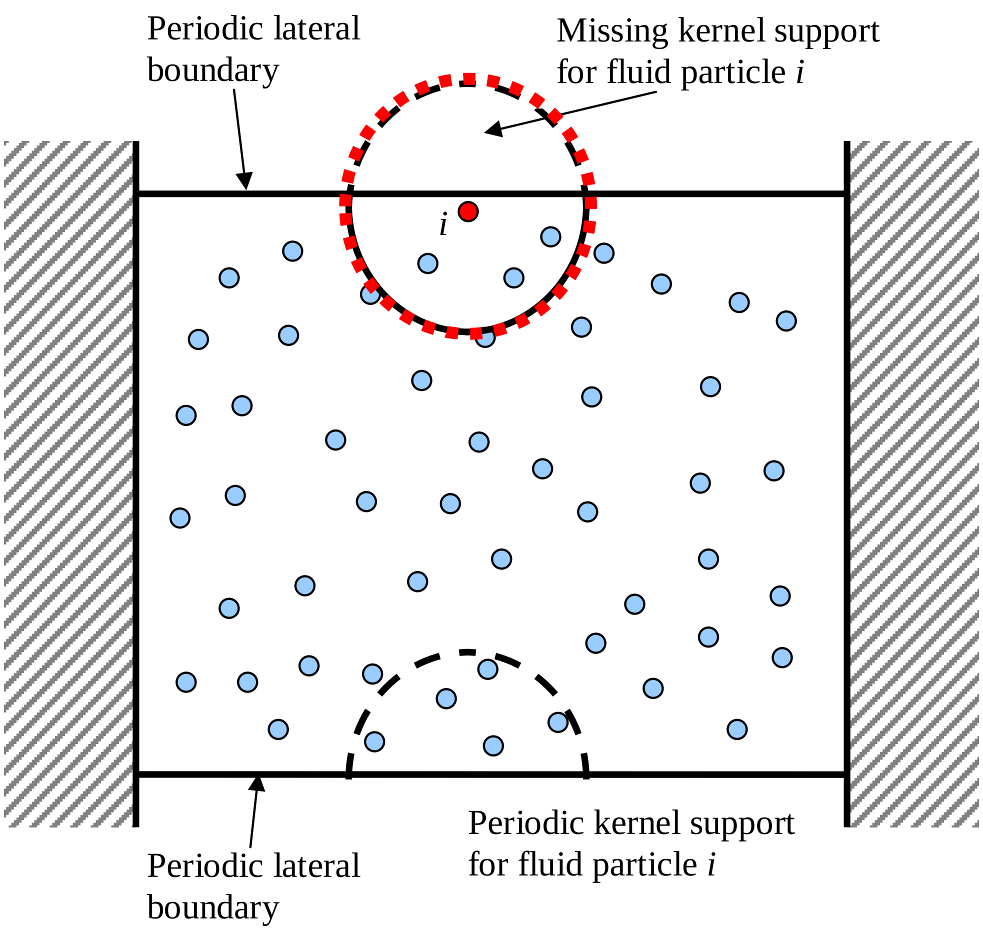
Particles cannot penetrate since they would cross through themselves!

Disadvantages: **Extremely difficult for any geometry other than a straight wall**

Corners (and therefore vorticity) very difficult to get right

Solid Wall Boundary Conditions

PERIODIC OPEN BOUNDARIES



Particles near an open lateral boundary interact with the particles near the complementary open lateral boundary on the other side of the domain.

Area of influence for the particle extends beyond top lateral boundary and is continued through periodic bottom boundary

Floating objects

- The object is considered ad as rigid body.

- **The force on each boundary particle is calculated as the sum of the contributions of the water particles at a distance of the kernel length.**

$$\mathbf{f}_k = \sum_{a \in WPs} \mathbf{f}_{ka}$$

Movement of the body

$$M \frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} = \sum_{k \in BPs} m_k \mathbf{f}_k \quad \text{and} \quad I \frac{\mathrm{d}\mathbf{\Omega}}{\mathrm{d}t} = \sum_{k \in BPs} m_k \left(\mathbf{r}_k - \mathbf{R}_0\right) \times \mathbf{f}_k$$

Each boundary particle that describes the moving object has velocity given by:

$$\mathbf{u}_k = \mathbf{V} + \mathbf{\Omega} \times \left(\mathbf{r}_k - \mathbf{R}_0\right)$$

Smoothed Particle Hydrodynamics

DISADVANTAGES comparing with other mesh-based CFD codes:

- ✓ The **interpolation method** used in SPH is very simple and it will be strongly affected by **particle disorder**. SPH gives reasonable results for the first order gradients, but they can be worse for higher order derivatives.

- ✓ **Turbulence treatment** is still an open field and more research is needed.

- ✓ **Boundary condition** implementation is a hard task and fluid particles penetration into boundaries must be avoided. There is no unanimity to choose the best boundary conditions approach.

- ✓ **Computation time is expensive** compared with other meshbased methods or CFD software.

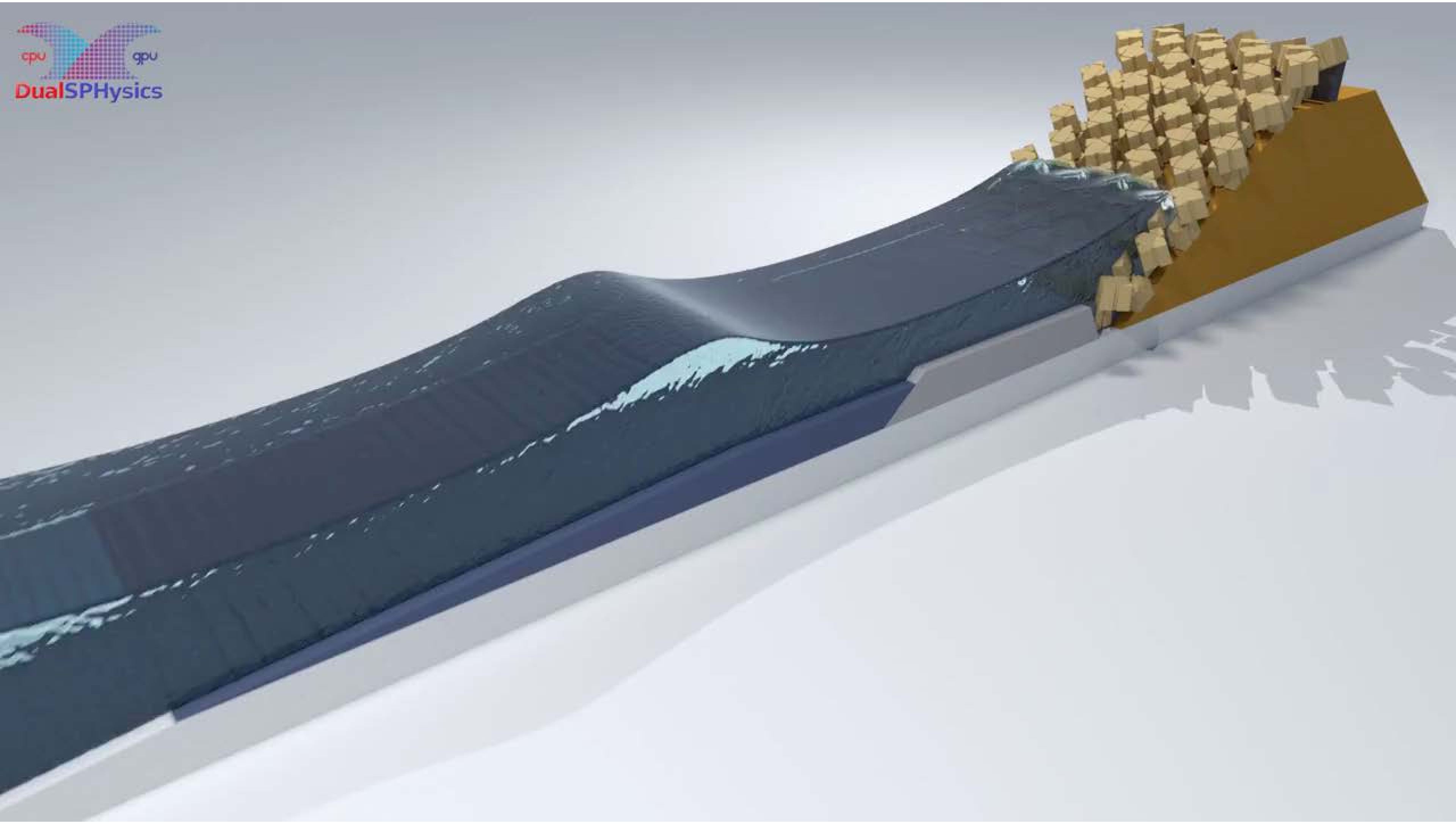
Smoothed Particle Hydrodynamics

ADVANTAGES comparing with other mesh-based CFD codes:

- ✓ Efficient treatment of the **large deformation** of free surfaces since there is no mesh distortion and **no need for a special treatment of the surface**
- ✓ Handling **complex boundary** evolution
- ✓ Distinguishing **between phases** due to holding material properties at each individual particle
- ✓ Capable of being **coupled with other** mesh dependent and meshless techniques

Smoothed Particle Hydrodynamics

ADVANTAGES comparing with other mesh-based CFD codes:



Smoothed Particle Hydrodynamics

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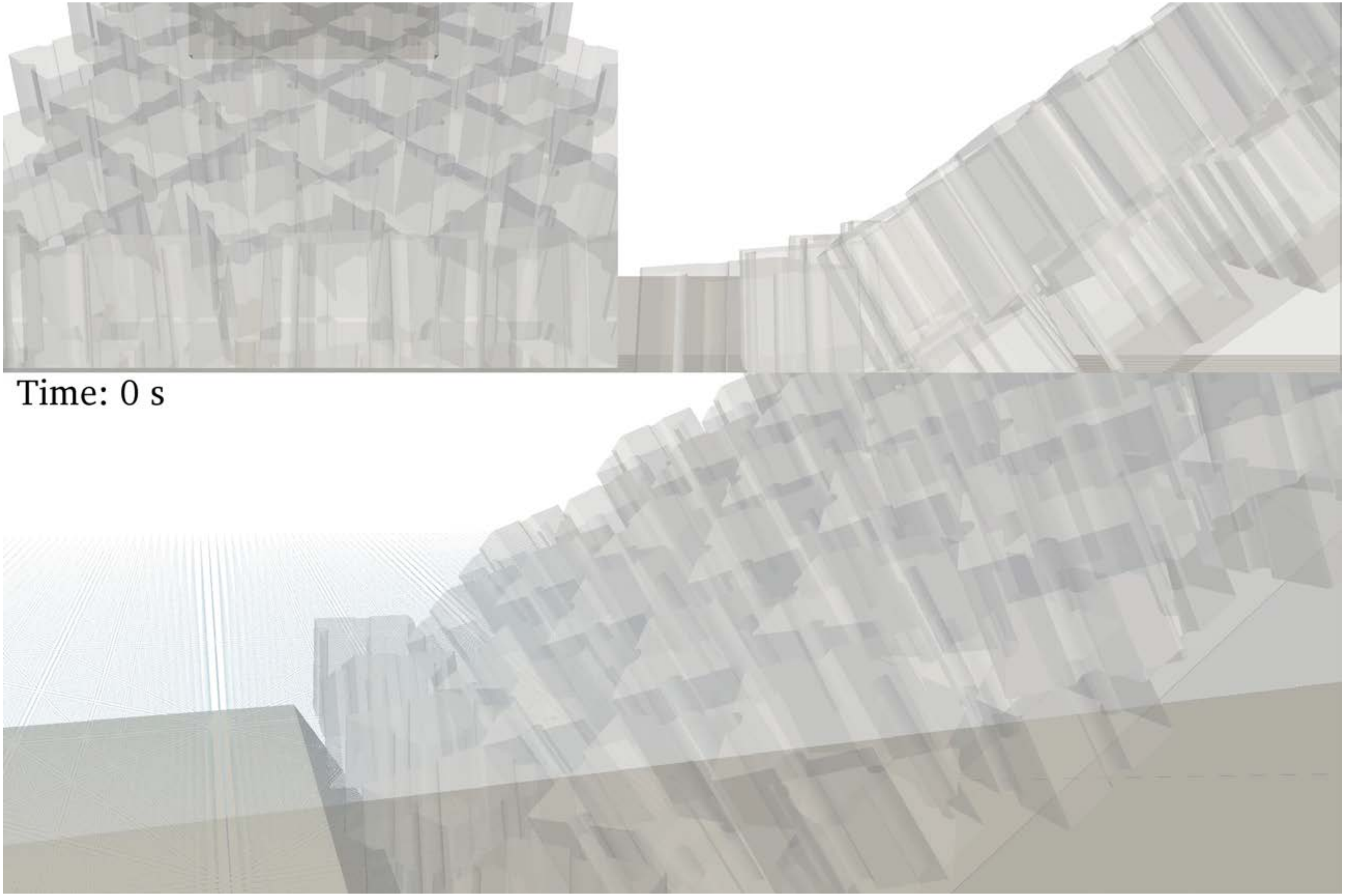

Smoothed Particle Hydrodynamics

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Smoothed Particle Hydrodynamics

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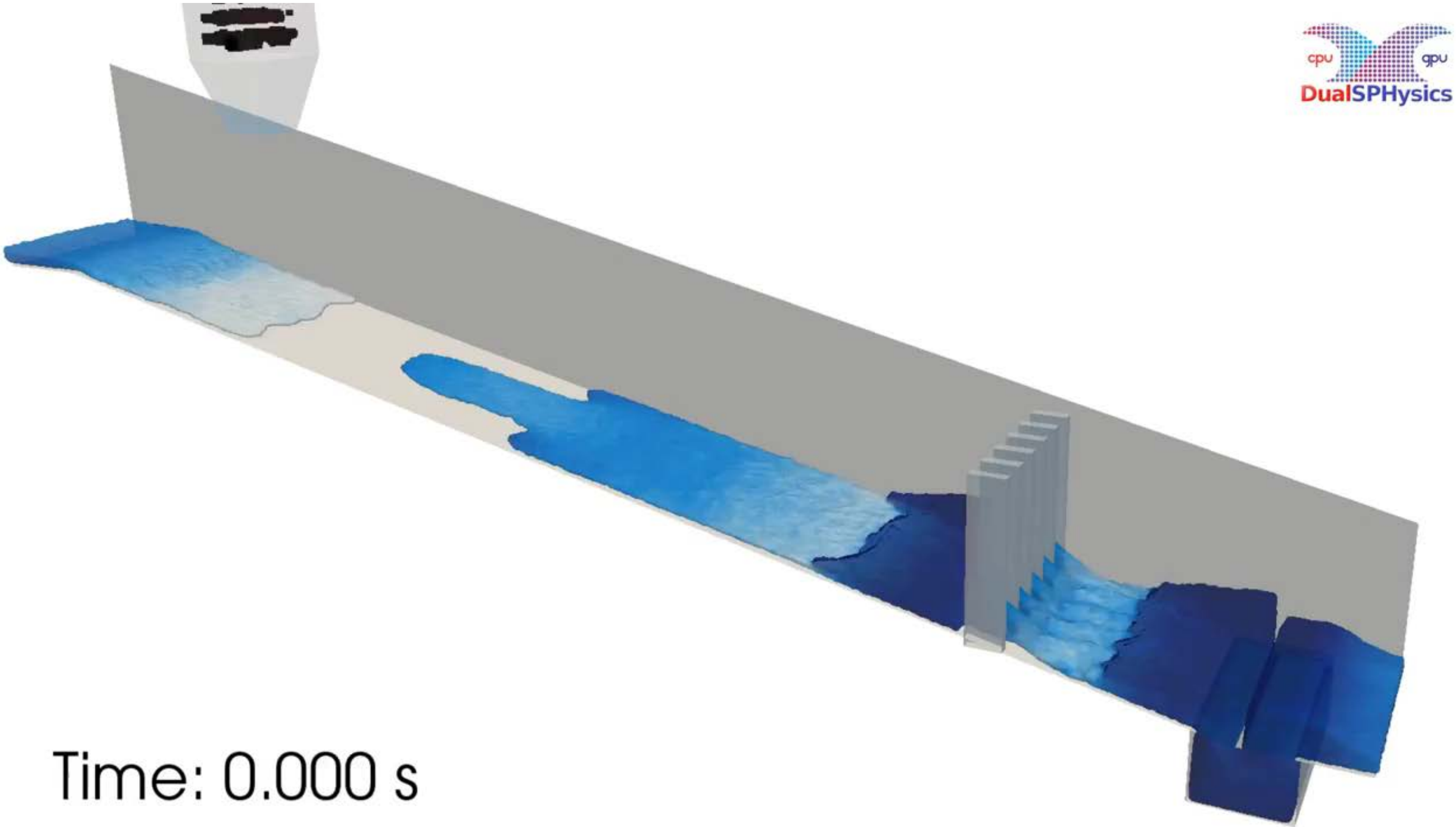

Can it be simulated with another numerical method???

Thanks for your attention

cpu gpu DualSPHysics

DualSPHysics

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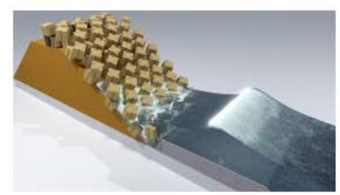
DualSPHysics is based on the Smoothed Particle Hydrodynamics model named SPHysics (www.sphysics.org).

The code is developed to study free-surface flow phenomena where Eulerian methods can be difficult to apply, such as waves or impact of dam-breaks on off-shore structures. **DualSPHysics** is a set of C++, CUDA and Java codes designed to deal with real-life engineering problems.



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